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LETTER TO THE EDITOR

The exact solution of a non-planar Ising model

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Abstract. A generalisation of the star–triangle transformation is used to establish a correspondence between a non-planar Ising lattice with two interaction constants and the standard triangular Ising model. The critical point of the non-planar lattice is obtained exactly for various values of the ratio of the interaction constants. The singularity in the zero-field free energy of the non-planar lattice is investigated. It is shown to be of the same form as that displayed by the corresponding standard Ising model.

The problems encountered and progress made in the solution of non-planar Ising models are well known (Green and Hurst 1964, Temperley 1972, Baxter 1982). In this article, a generalisation of the star–triangle transformation (Onsager 1944, Wannier 1945, Fisher 1959)—which will be called the $K(3,3)$–$K(3)$ transformation—is used to solve the Ising lattice of figure 1. The notation here is that standard in graph theory (Essam and Fisher 1970). Thus $K(3,3)$ is the complete bichromatic graph shown in figure 2 (there is, of course, no vertex at the centroid here) whilst $K(3)$ is the complete graph on three vertices (the triangle). The non-planarity of the lattice in figure 1 follows directly from the non-planarity of $K(3,3)$ (Essam and Fisher 1970).

The (zero-field) partition function associated with the graph $K(3,3)$ in figure 2 is a sum of terms of the form

$$\exp[K_1(\sigma_1\sigma_4+\sigma_4\sigma_2+\sigma_2\sigma_5+\sigma_5\sigma_3+\sigma_3\sigma_6+\sigma_6\sigma_1)+K_2(\sigma_1\sigma_5+\sigma_2\sigma_6+\sigma_3\sigma_4)], \qquad (1)$$

where $K_1=\beta J_1$, $K_2=\beta J_2$ and the notation is standard. The interaction constants J_1 and J_2 correspond respectively to first and second neighbour interactions. If one sums

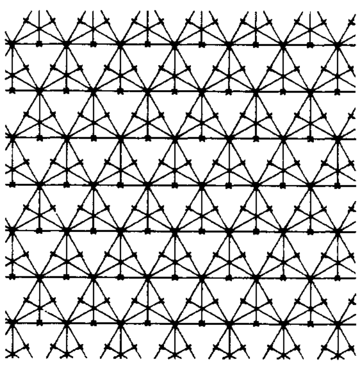

Figure 1. A non-planar lattice.

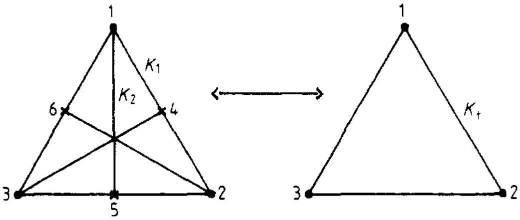

Figure 2. The $K(3,3)$–$K(3)$ transformation.

(1) over all values of σ_4, σ_5 and σ_6, the result can be written in the form

$$8\cosh[(\sigma_1+\sigma_2)K_1+\sigma_3K_2]\cosh[(\sigma_2+\sigma_3)K_1+\sigma_1K_2]\cosh[(\sigma_3+\sigma_1)K_1+\sigma_2K_2]. \qquad (2)$$

For given K_1 and K_2, the expression (2) has only two distinct values for $\sigma_1=\pm1$, $\sigma_2=\pm1$, and $\sigma_3=\pm1$. These are

$$8\cosh^3(2K_1+K_2) \qquad (\text{for } \sigma_1=\sigma_2=\sigma_3),$$
$$8\cosh(2K_1-K_2)\cosh^2 K_2 \qquad (\text{otherwise}). \qquad (3)$$

The (zero-field) partition function of the triangle in figure 2—augmented by a factor Δ—is a sum of terms of the form

$$\Delta\exp[K_t(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)], \qquad (4)$$

where the notation is again standard. For given Δ and K_t, the expression (4) again has only two distinct values for $\sigma_1=\pm1$, $\sigma_2=\pm1$, and $\sigma_3=\pm1$. These are

$$\Delta\exp(3K_t) \qquad (\text{for } \sigma_1=\sigma_2=\sigma_3), \qquad \Delta\exp(-K_t) \qquad (\text{otherwise}). \qquad (5)$$

The compatibility between (3) and (5) allows one to identify (2) and (4) for all spin states provided that K_1, K_2 and Δ, K_t satisfy

$$\Delta\exp(3K_t)=8\cosh^3(2K_1+K_2), \qquad \Delta\exp(-K_t)=8\cosh(2K_1-K_2)\cosh^2 K_2. \qquad (6)$$

These equations may be solved for Δ and K_t in terms of K_1 and K_2. This gives

$$K_t=\tfrac{1}{4}\ln\{\cosh^3(2K_1+K_2)/[\cosh(2K_1-K_2)\cosh^2 K_2]\}, \qquad (7)$$
$$\Delta=8\cosh^{3/4}(2K_1+K_2)\cosh^{3/4}(2K_1-K_2)\cosh^{3/2} K_2.$$

These results (or equivalently those of (6)) are the basic equations of the $K(3,3)$–$K(3)$ transformation.

Suppose that the lattice in figure 1 has $4N$ sites and that the interaction constants J_1 and J_2—ascribed in the obvious way—are each uniform in value over the entire lattice. The partition function of the lattice can then be written $Z_{4N}(K_1,K_2)$. This object is a sum over all spin states of a product of terms of the form (1)—one for each of the embeddings of $K(3,3)$ made appropriate by figure 1. The spins on all the lattice sites equivalent to the sites 4, 5 and 6 in figure 2 may be summed over, thus reducing each of the above terms to the form (2). By imposing equations (7), one may then rewrite each of these terms in the form (4). It then remains to sum over the spins on all the lattice sites equivalent to the sites 1, 2 and 3 in figure 2. These constitute a triangular Ising lattice of N sites and interaction parameter K_t. Thus

$$Z_{4N}(K_1,K_2)=\Delta^N Z_N^t(K_t). \qquad (8)$$

Here, K_t and Δ are as in (7) and Δ appears raised to the power N since this is the appropriate number of embeddings of $K(3,3)$ in the original lattice. The quantity $Z^t_N(K_t)$ is the partition function of a triangular Ising lattice of N sites and interaction parameter K_t. Since this is well known (Green and Hurst 1964, Temperley 1972, Baxter 1982), (7) and (8) provide an exact solution of our non-planar lattice.

Interest centres on the variation of the properties of the lattice in figure 1 with temperature. Now J_1 and J_2 are constant. Let r denote the ratio J_2/J_1. (The excluded case $J_1=0$ is trivial and can be dealt with directly.) This allows (7) and (8) to be used, in practice, with $K_1=K$ and $K_2=rK$ and the dependence on K (which is a dimensionless inverse temperature) to be studied. Since J_1 and J_2 correspond to first and second neighbour interactions, attention here will be restricted to the interval $0 \leqslant r \leqslant 1$. (The discussion of the complexities of competing ferromagnetic and antiferromagnetic interactions is inappropriate in this first account and will be left to another occasion.)

An issue of particular interest is the location of the critical point. For fixed values of r in the interval considered, (7) yields a one–one correspondence between K and K_t. Hence the non-planar lattice has a unique critical point K_c inherited from the triangular lattice. The critical point of the triangular lattice occurs at $K_t=\frac{1}{4}\ln 3$ (Baxter 1982). Hence K_c is the value of K which satisfies

$$\cosh^3[(2+r)K]/\{\cosh[(2-r)K]\cosh^2 rK\}=3. \tag{9}$$

For the extreme values of r considered here, (9) yields

$$\cosh 2K_c=\surd 3 \quad \text{or} \quad K_c=0.573\,1079\ldots \quad (r=0)$$
$$\cosh 2K_c=\tfrac{1}{2}(1+3^{1/3}) \quad \text{or} \quad K_c=0.326\,6682\ldots \quad (r=1). \tag{10}$$

Clearly, $r=0$ corresponds to a decorated triangular lattice (Syozi 1951, Naya 1954), which provides a check on our results, whilst $r=1$ corresponds to the case in which the first and second neighbour interactions in figure 1 are equal. For other values of r, (9) can be solved numerically. Results obtained in this way are presented in figure 3. The value of K_c decreases as r increases which one might expect on general grounds (Griffiths 1972) since K_c^{-1} is a dimensionless critical temperature.

Another issue of interest is the nature of the singularity at the critical point. From (8) it follows that the free energy per spin of the non-planar lattice

$$F(K_1,K_2)=-\tfrac{1}{4}kT\ln\Delta+\tfrac{1}{4}F_t(K_t), \tag{11}$$

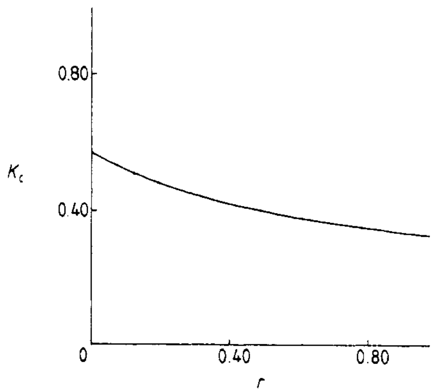

Figure 3. The critical point K_c for various interaction ratios r.

where F_t is the free energy per spin of the corresponding triangular lattice and the notation is standard. Now $\ln \cosh x$ is an analytic function of x. (It is, of course, essentially the free energy of a single Ising bond at dimensionless inverse temperature x.) Thus, with $K_1 = K$ and $K_2 = rK$, one finds that K_t and $\ln \Delta$ are both analytic functions of K. This means that F inherits its singular behaviour directly from F_t. Moreover, one can show directly that, at least for the values of r which concern us,

$$K_t = \tfrac{1}{4} \ln 3 + A(K - K_c) + \ldots \tag{12}$$

with A non-zero. (The correspondence between K and K_t is one–one.) It thus follows rigorously that, leaving aside amplitudes, our non-planar lattice has the same 'critical behaviour' (and critical exponents α and α') as the triangular lattice. This is, of course, consistent with what one would expect from universality.

The analysis presented in this article can be extended to the case in which the spins 4, 5 and 6 in figure 2 are allowed to interact directly in pairs. The appropriate embeddings of $K(3, 3)$ in figure 1 can then be replaced, at the cost of complications which are best avoided here, by embeddings of this new graph.

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